Differential equation and convergence

An investment grows according to the following differential equation:

$$\frac{dA}{dt} = r \cdot A + k$$

where $A(t)$ is the capital growth at time t, and r and k are constants. Find the general solution of $A(t)$. Establish the conditions that must be satisfied for the solution to be convergent, justifying your answer.

Solution

We rewrite:

$$dA = (r \cdot A + k)dt$$

$$\frac{dA}{rA + k} = dt$$

$$\frac{\ln(rA + k)}{r} = t + C$$

Solving for A:

$$rA + k = e^{rt+rC}$$

$$A = \frac{e^{rt+rC} - k}{r}$$

$$A = C_1 e^{rt} - \frac{k}{r}$$

For $A(t)$ to be convergent as $t \to \infty$, it is necessary that $A(t)$ approaches a finite value. We observe the behavior of the term Ce^{rt}. If $r > 0$, then $e^{rt} \to \infty$ as $t \to \infty$, and $A(t)$ diverges. For convergence, we need $r < 0$.